

# Higher-order corrections to the relativistic perihelion advance and the mass of binary pulsars.

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We study the general relativistic orbital equation and using a straightforward perturbation method and a mathematical device first introduced by d'Alembert, we work out approximate expressions of a bound planetary orbit in the form of trigonometrical polynomials and the first three terms of the power series development of the perihelion advance. The results are applied to a more precise determination of the total mass of the double pulsar J0737-3039.

## I. INTRODUCTION

The general relativistic orbital equation for a planet revolving around a star is deduced from the Schwarzschild line element

$$ds^2 = c^2 \gamma dt^2 - \gamma^{-1} dr^2 - r^2 d\Omega^2, \quad (1)$$

$$\gamma = 1 - 2r^*/r, \quad d\Omega^2 = d\theta^2 - \sin^2\theta d\varphi^2,$$

where  $r^* = \mu/c^2$  and  $\mu = GM$  are the gravitational radius and the standard gravitational parameter of the star, respectively. All remaining symbols have their usual meaning for this type of problem. According to the geodesic hypothesis, the path followed by the planet, considered as a test-body not to disturb the metric, can be determined using the time-like Lagrangian

$$2L = \left(\frac{ds}{d\tau}\right)^2 = c^2 \gamma \dot{t}^2 - \gamma^{-1} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2), \quad (2)$$

and a variational principle that uses the functional

$$S[q] = \int_{\tau_1}^{\tau_2} L(q, \dot{q}) d\tau, \quad \dot{q} = dq/d\tau, \quad (3)$$

where  $\tau$  is the planet's proper time and the function  $q = q(\tau)$  collectively denotes the degrees of freedom of a possible generic planetary motion. The path actually followed is what makes  $S[q(\tau)]$  stationary, i.e. the functional derivative

$$\frac{\delta S[q(\tau')]}{\delta q(\tau)} = \int_{\tau_1}^{\tau_2} \frac{\delta L[q(\tau'), \dot{q}(\tau')]}{\delta q(\tau)} d\tau = 0, \quad (4)$$

is zero when computed on the effective motion. From the working point of view it is well known that a cancelation, in the integrand, of the functional derivative is equivalent to that of the Euler-Lagrange derivative

$$\frac{\delta}{\delta q} = \frac{\partial}{\partial q} - \frac{d}{d\tau} \frac{\partial}{\partial \dot{q}}, \quad (5)$$

and so we obtain four second-order differential equations (the Euler-Lagrange equations) determining the sought time-like geodesic

$$\frac{\partial L}{\partial q} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} = 0, \quad q = r, \theta, \varphi, t, \quad \dot{q} = dq/d\tau. \quad (6)$$

It is worth noting that from the defining equation of the proper time,  $ds/d\tau = c$ , we have  $2L = c^2$ , namely the Lagrangian is a constant of the motion.

We observe that the Schwarzschild metric, and the Lagrangian, are invariant under the reflection

$$(t, r, \theta, \varphi) \mapsto (t, r, \pi - \theta, \varphi), \quad (7)$$

at the hyperplane  $\theta = \pi/2$ , so the mirror image of a geodesic curve clearly has the same property. In particular, if we consider a geodesic which at  $\tau = 0$  starts within the symmetry hyperplane and is tangent to it, it must coincide with the transformed geodesic, since the initial values of position and velocity determine a geodesic uniquely. These considerations are confirmed by the analysis of the Euler-Lagrange equation for  $\theta$

$$\ddot{\theta} + \frac{2\dot{r}}{r} \dot{\theta} - \frac{\dot{\varphi}^2}{2} \sin 2\theta = 0, \quad (8)$$

which admits the solution  $\theta = \pi/2$  satisfying the initial conditions  $\theta_0 = \pi/2$ ,  $\dot{\theta}_0 = 0$ . If we reorient the coordinate system so that these conditions are met, the motion takes place in the equatorial plane, and we can simplify the Lagrangian (2) assuming  $\sin\theta = 1$ ,  $\dot{\theta} = 0$

$$2L = c^2 \gamma \dot{t}^2 - \gamma^{-1} \dot{r}^2 - r^2 \dot{\varphi}^2. \quad (9)$$

The coordinates  $\varphi$  and  $t$  are both cyclical: they appear in the Lagrangian only in dotted form, and this means two conservation laws. For the azimuth  $\varphi$  we have

$$-\frac{d}{d\tau} \frac{\partial L}{\partial \dot{\varphi}} = 0 \Rightarrow -\frac{\partial L}{\partial \dot{\varphi}} = r^2 \dot{\varphi} = h = \text{const.}, \quad (10)$$

while for  $t$  we have

$$\frac{d}{d\tau} \frac{\partial L}{\partial \dot{t}} = 0 \Rightarrow \frac{\partial L}{\partial \dot{t}} = c^2 \gamma \dot{t} = \kappa = \text{const.} \quad (11)$$

The two constants are related to the angular momentum and to the energy, respectively. Insertion of the two integrals into Eq. (9) leads to

$$2L = \gamma^{-1} \frac{\kappa^2}{c^2} - \gamma^{-1} \dot{r}^2 - \frac{h^2}{r^2}. \quad (12)$$

In place of the Euler-Lagrange equation for  $r$ , it is easier to derive the radial equation from the integral  $2L = c^2$ , obtaining so after multiplication by  $\gamma$

$$\dot{r}^2 + \frac{h^2}{r^2} - \frac{2\mu}{r} - \frac{2h^2 r^*}{r^3} + \left(c^2 - \frac{\kappa^2}{c^2}\right) = 0. \quad (13)$$

We need to cast the path equation into a form which clearly displays the fact that we are dealing with a Keplerian orbit subjected to small relativistic corrections, so we find it convenient to eliminate the variable  $\tau$  and introduce the angle  $\varphi$  instead. This is possible since  $\tau$  does not enter the equation directly, but only via its differential  $d\tau$ , so we transform the  $\tau$ -derivative into a  $\varphi$ -derivative, according to the identity  $d/d\tau = (h/r^2)d/d\varphi$ . Moreover, the formulas simplify considerably if we replace  $r$  by its reciprocal  $u = 1/r$ . Denoting by a prime differentiation with respect to  $\varphi$ , we have  $\dot{r}^2 = h^2 u'^2$  and Eq. (13) becomes

$$u'^2 + u^2 - \frac{2\mu}{h^2}u - 2r^*u^3 + \frac{1}{h^2}\left(c^2 - \frac{\kappa^2}{c^2}\right) = 0, \quad (14)$$

while with a differentiation with respect to  $\varphi$  we find

$$u'(u'' + u - \frac{\mu}{h^2} - 3r^*u^2) = 0. \quad (15)$$

A solution of this equation is  $u' = 0$ , or  $u = \text{const.}$ , and this means a circular orbit. Ruling out this possibility, Eq. (15) requires that

$$u'' + u = \frac{\mu}{h^2} + 3r^*u^2. \quad (16)$$

Comparing this equation with the classical Binet's formula for a particle subjected to a central force of magnitude  $f(u)$  in polar coordinates

$$u'' + u = -\frac{f(u)}{h^2 u^2}, \quad (17)$$

it appears that a particle moving in the Schwarzschild field will behave as though it were under the influence of an effective Newtonian inverse-square force plus an additional fourth-power inverse force

$$-\left(\frac{\mu}{r^2} + \frac{3r^*h^2}{r^4}\right)\frac{\mathbf{r}}{r}, \quad (18)$$

in a framework in which proper time is used as independent variable. We remark that for planetary trajectories, in Eq. (16) the terms  $u$  and  $\mu/h^2$  are comparable, while  $u$  and  $3r^*u^2$  differ by a factor  $3r^*u$ . The maximum value of this quantity corresponds to the planet which is nearest to the star, so, for Mercury, with  $r^* \approx 1.476 \cdot 10^5 \text{cm}$ ,  $r \approx 5.5 \cdot 10^{12} \text{cm}$ , it is  $3r^*/r \approx 10^{-7}$ . This implies that Eq. (16) represents in general an oscillator with a constant forcing term and a weak quadratic nonlinearity which affects its frequency<sup>16</sup>.

The orbital equation (16) is not the complete solution to motion problem, which would require the knowledge

of the function  $\varphi(t)$ , being  $t$  the coordinate time, which is the time measured by an observer at rest at great distance from the origin, and therefore the quadrature of the equations

$$\frac{d\tau}{d\varphi} = \frac{1}{hu^2}, \quad \frac{dt}{d\varphi} = \frac{\kappa}{\gamma hu^2}, \quad (19)$$

but this does not concern us here.

The importance of Eq. (16) is due to a variety of reasons. It originates, without any approximation, from an exact solution of Einstein's equation. Because of the lacking of a time variable, it is entirely geometric, so it can be employed to deduce the precise shape of the orbit and one of the most important post-Newtonian predictions of the theory: the advance of the perihelion of an elliptical orbit<sup>7</sup>. The equation, then, is naturally linked to Newtonian dynamics, that in the description of a planetary motion provides already an excellent approximation as well as a clear connection with observation. Last, it is simple when compared with other post-Newtonian equations of planetary dynamics, and therefore its mathematical and theoretical limits can be clearly defined. We will use it to calculate the orbit of a test particle in the gravitational field external to a non-spinning spherical mass eventually to the order  $1/c^n$  for any arbitrary positive integer  $n$ , and the corresponding formulas of the periastron advance to the same order. If the effects predicted fall in the range of the observability, measurability or indirect determination for those physical systems where the equation is applicable, then no doubt they should necessarily be taken into account in all specific cases.

A determination of higher-order terms of the periastron advance of a binary pulsar by using the second post-Newtonian (2PN) method have been effected by some authors<sup>5,18</sup>. Their results based on the 2PN theory can be applied to the problem of a test particle under a strong gravitational field by letting the mass of one pulsar theoretically approach zero. Since one can argue about the rigor of this and other methods used to handle the 2PN problem of motion, we think that a proper analysis of the correctness and of the limits of accuracy of these approaches should be based on the agreement under some convenient limit with the results exposed in the present paper. When considering relativistic effects, other post-Keplerian phenomena come into play in the motion of bodies. For example, one could insert the relativistic force (18) as a perturbing radial acceleration in Gauss equations for the variations of the Keplerian orbital elements<sup>1</sup>. In particular, in the expression of the time derivative of the mean anomaly, in addition to the radial force it appears explicitly the motion of the perihelion. We could cite at this regard two recent works<sup>10,11</sup> on the secular advance of the mean anomaly in binary systems, which could be easily extended to include the higher-order effects calculated in this paper. It would also be interesting to compare the results so obtained with those computed by means of the post-Newtonian Lagrangian planetary equations<sup>3</sup>, and this could be the

subject of future work.

## II. THE ITERATION METHOD

In the following we will express  $h$  in terms of elliptic elements of Newtonian approximation, so  $h = \sqrt{\mu p}$ , where  $p = a(1 - e^2)$  is the semi-latus rectum,  $a$  is the semi-major axis and  $0 \leq e < 1$  is the eccentricity<sup>6</sup>. Although we have excluded before the circular orbit, the orbital equation encompasses also this possibility. To make its structure more apparent and to facilitate the calculations, we cast Eq. (16) in dimensionless form, which represents a well-known problem in mathematical physics

$$u'' + u = 1 + \epsilon u^2, \quad (20)$$

where this time  $u$  means  $p/r$  and  $\epsilon$  is the pure number  $3r^*/p$ . Thus, when  $r = a(1 \pm e)$ ,  $u = 1 \mp e$ . To find an approximate solution to this equation, one could use regular perturbation theory, writing  $u$  in the form of a perturbative expansion in powers of  $\epsilon$ , but it runs in trouble here, since we wish an expansion that converges for all values of the independent variable  $\varphi$ . To do this in the most convenient way, we perform a shift in the origin  $w = u - 1$ , in order to write the equation as that of the perturbed harmonic oscillator

$$w'' + w = \epsilon(1 + w)^2. \quad (21)$$

When  $\epsilon = 0$  the equation becomes

$$w''_0 + w_0 = 0, \quad (22)$$

whose general solution is

$$w_0(\varphi) = w_0(0) \cos \varphi + w'_0(0) \sin \varphi. \quad (23)$$

We shall adopt the initial conditions  $w_0(0) = e$ ,  $w'_0(0) = 0$ , which hereafter we shall denote standard, and so

$$w_0(\varphi) = e \cos \varphi. \quad (24)$$

This choice of the initial conditions leads to a simplification of the algebra. Solution (24) has frequency one and period  $2\pi$ . The equation of the orbit is then

$$\frac{w_0 + 1}{p} = \frac{1}{r} = \frac{1 + e \cos \varphi}{p}, \quad (25)$$

that is an ellipse, where the angle  $\varphi = 0$  locates the position of the perihelion (because there the function  $w_0$  is maximal), and identifies the direction of the apse line, the greatest symmetry axis of the orbit in the plane.

If we try a straightforward iterative perturbation scheme to solve Eq. (21) attaching the appropriate subscripts (the iteration numbers) to its sides, for the first-order solution  $w_1$  we obtain the equation

$$w''_1 + w_1 = \epsilon(1 + w_0)^2, \quad (26)$$

of a forced pendulum where there is a resonant  $\cos \varphi$  term. If we do not want that the pendulum oscillates with an ever increasing period ( $w_1$  must stay small for all values of  $\varphi$ ), then the external force is not allowed to have a Fourier component with the same periodicity as the pendulum itself. Note here and in the following that, according to the method of undetermined coefficients, the particular solution of an equation of the form

$$w'' + k^2 w = \sum_n X_n \cos nk\varphi, \quad (27)$$

is formally expressed by

$$w = \frac{X_0}{k^2} + \left[ \frac{X_n \cos nk\varphi}{k^2(1 - n^2)} \right]_{n=1} + \sum_{n>1} \frac{X_n \cos nk\varphi}{k^2(1 - n^2)}, \quad (28)$$

whose second term is singular, but can be regularized since, applying the L'Hospital's rule, we have

$$\lim_{n \rightarrow 1} \frac{X_n \cos nk\varphi}{k^2(1 - n^2)} = \frac{X_1 \varphi \sin k\varphi}{2k}, \quad (29)$$

with  $\varphi$  explicitly present outside the argument of the trigonometrical function, and therefore growing without limits with time  $t$ , since  $\varphi$  is a monotonic function of  $t$ . Obviously this would destroy the stability of the orbit. Therefore this simple perturbation scheme does not work for Eq. (21). Things go differently if we rearrange Eq. (21) in the form

$$w'' + k_\epsilon^2 w = \epsilon(1 + w^2), \quad k_\epsilon^2 \equiv 1 - 2\epsilon. \quad (30)$$

The equation is unchanged, but now we have chosen to look at the isolated linear term on the right side as part of the unperturbed equation, which is

$$w''_0 + k_\epsilon^2 w_0 = 0. \quad (31)$$

Equation (31) now represents an oscillator whose natural frequency is  $k_\epsilon$ , smaller than 1, and its solution obeying to the standard initial conditions is

$$w_0(\varphi) = e \cos k_\epsilon \varphi. \quad (32)$$

It follows that, with respect to the Newtonian oscillator (22), in the relativistic case, to the lowest order, the values of  $r$ , which trace out an approximated ellipse, do not begin to repeat until somewhat after the radius vector has made a complete revolution. Hence the orbit may be regarded as being an ellipse which is slowly rotating. In particular, the angular advance for revolution of the apse line is given by

$$\Delta\omega = 2\pi \left( \frac{1}{k_\epsilon} - 1 \right) = 2\pi\epsilon + O(\epsilon^2). \quad (33)$$

This is Einstein's perihelion formula, but it represents only the first term of a series development in powers of  $\epsilon$ . Our aim is to compute this series up to order  $\epsilon^3$ . Although the first approximation gives a satisfactory degree

of accuracy for ordinary planetary problems, and the second one can cope with particular astrophysical situations, we push a step further the computation because with little extra work we shall exhaust all conceivable theoretical needs before the orbital equation fails to represent the relativistic motion of bodies in strong-field situations.

Let us try again the perturbation scheme on Eq. (30). We have now

$$w_1'' + k_\epsilon^2 w_1 = \epsilon(1 + w_0^2), \quad (34)$$

$$= \epsilon \left( 1 + \frac{1}{2} e^2 \right) + \epsilon \frac{1}{2} e^2 \cos 2k_\epsilon \varphi, \quad (35)$$

that we solve with the standard initial conditions. It is to be noticed that this time we have not any more the resonant term  $\cos k_\epsilon \varphi$ . This equation is of the type

$$w_1'' + k_\epsilon^2 w_1 = A + B \cos 2k_\epsilon \varphi. \quad (36)$$

The solution is the sum of the general solution of the homogeneous part and of a particular integral of the complete equation. This integral can be found using formula (28). Thus the solution of Eq. (36) is

$$w_1 = E \cos k_\epsilon \varphi + A k_\epsilon^{-2} - \frac{B k_\epsilon^{-2}}{3} \cos 2k_\epsilon \varphi. \quad (37)$$

As  $w_1$  is of order  $\epsilon$  as  $A, B$  are, we put  $k_\epsilon^{-2} = 1$ , while the constant  $E$  is determined by the standard initial conditions to be

$$E = e - \epsilon \left( 1 + \frac{e^2}{3} \right), \quad (38)$$

and so we find

$$w_1 = \epsilon \left( 1 + \frac{1}{2} e^2 \right) + \left[ e - \epsilon \left( 1 + \frac{1}{3} e^2 \right) \right] \cos k_\epsilon \varphi - \epsilon \frac{1}{6} e^2 \cos 2k_\epsilon \varphi, \quad (39)$$

which we shall write in abridged form

$$w_1 = A_1 + (e + E_1) \cos k_\epsilon \varphi + B_1 \cos 2k_\epsilon \varphi, \quad (40)$$

where as a notational aid we agree, here and in the following, that the subscript number  $i$  attached to a capital letter representing a coefficient wants to emphasize the presence of the factor  $\epsilon^i$ . This solution is periodical and bounded for all values of  $\varphi$ , and represents the general relativistic first-order deviation from the classical elliptical orbit. In the next iteration, which should give the solution correct to order  $\epsilon^2$

$$w_2'' + k_\epsilon^2 w_2 = \epsilon(1 + w_1^2), \quad (41)$$

the right-hand side will present again a resonant term, since to this order we get

$$w_2'' + k_\epsilon^2 w_2 = A + H_2 \cos k_\epsilon \varphi + B \cos 2k_\epsilon \varphi + C \cos 3k_\epsilon \varphi, \quad (42)$$

$$\text{where } H_2 \equiv \epsilon e(2A_1 + B_1) = \epsilon^2 \frac{e(12 + 5e^2)}{6}, \quad (43)$$

with  $A, B, C$  to be written later. To get rid of the resonant term  $H_2 \cos k_\epsilon \varphi$  on the right side of Eq. (42) we will use a device which stems naturally from the logical path we followed in writing Eq. (30) and that was introduced for the first time by d'Alembert to control the plague of the secular terms present in the integration of the equation of the lunar motion, which is of the same type of that we are considering here<sup>4</sup>. In order to suppress the unwanted cosine term, we add a counter term and write Eq. (30) in the form

$$w'' + \left( k_\epsilon^2 - \frac{H_2}{e} \right) w = \epsilon(1 + w^2) - \frac{H_2}{e} w, \quad (44)$$

and consequently we consider the approximate equation

$$w_2'' + k_\epsilon^2 w_2 = \epsilon(1 + w_1^2) - \frac{H_2}{e} w_1, \quad (45)$$

$$k_\epsilon^2 \equiv k_\epsilon^2 - \frac{H_2}{e}. \quad (46)$$

In the right side, the added term suppresses the resonant term, since replacing  $w_1$  with  $e \cos k_\epsilon \varphi$  (the other terms of  $w_1$  would give terms of order greater than  $\epsilon^2$ ), we obtain just  $-H_2 \cos k_\epsilon \varphi$ . In the left side the coefficient  $k_\epsilon^2$  will be diminished by the amount  $H_2/e$ , so that we get

$$k_\epsilon^2 = 1 - 2\epsilon - \frac{12 + 5e^2}{6} \epsilon^2. \quad (47)$$

This way we shall obtain an acceptable solution  $w_2$  of Eq. (41) to order  $\epsilon^2$  and, at the same time, the correction to the frequency to the same order. The procedure can obviously be repeated until we have reached the required approximation degree for both solution and frequency. The essence of this method was rediscovered by Lindstedt<sup>14</sup> and its practical application was further elaborated by Poincaré<sup>16</sup>. Equation (45) has the form

$$w_2'' + k_\epsilon^2 w_2 = A + B \cos 2k_\epsilon \varphi + C \cos 3k_\epsilon \varphi, \quad (48)$$

$$A \equiv \frac{(2 + e^2)}{2} \epsilon - \frac{(3e + e^3)}{3} \epsilon^2, \quad (49)$$

$$B \equiv \frac{e^2}{2} \epsilon - \frac{3e + e^3}{3} \epsilon^2, \quad (50)$$

$$C \equiv -\frac{e^3}{6} \epsilon^2, \quad (51)$$

and its general solution, by Eq. (28), is

$$w_2 = E \cos k_\epsilon \varphi + A k_\epsilon^{-2} - \frac{B k_\epsilon^{-2}}{3} \cos 2k_\epsilon \varphi - \frac{C k_\epsilon^{-2}}{8} \cos 3k_\epsilon \varphi, \quad (52)$$

where now it suffices to put  $k_\epsilon^{-2} = 1 + 2\epsilon$ . We thus find, by determining  $E$  by means of the standard initial conditions, the second-order approximation to the orbit

$$w_2 = A_1 + A_2 + (e + E_1 + E_2) \cos k_\epsilon \varphi + (B_1 + B_2) \cos 2k_\epsilon \varphi + C_2 \cos 3k_\epsilon \varphi, \quad (53)$$

where

$$\begin{aligned} A_2 &= \epsilon^2 \frac{6 - 3e + 3e^2 - e^3}{3}, \\ E_2 &= \epsilon^2 \frac{29e^3 - 96e^2 + 96e - 288}{144}, \\ B_2 &= \epsilon^2 \frac{e^3 - 3e^2 + 3e}{9}, \\ C_2 &= \epsilon^2 \frac{e^3}{48}. \end{aligned}$$

Let us consider now the next equation

$$w_3'' + k_2^2 w_3 = \epsilon(1 + w_2^2) - \frac{H_2}{e} w_2. \quad (54)$$

Since we are interested only in the resonant term of  $O(\epsilon^3)$  we do not solve this equation, but we can extract quickly from the right side the secular generating term  $H_3 \cos k_2 \varphi$  (see Appendix), where

$$H_3 = \epsilon e(2A_2 + B_2) = \epsilon^3 \frac{e(36 - 15e + 15e^2 - 5e^3)}{9}. \quad (55)$$

Following d'Alembert's method, this term will be canceled writing Eq. (54) in the form

$$w_3'' + k_3^2 w_3 = \epsilon(1 + w_2^2) - \frac{H_2 + H_3}{e} w_2, \quad k_3^2 = k_2^2 - \frac{H_3}{e}, \quad (56)$$

and we have

$$k_3^2 = 1 - 2\epsilon - \frac{12 + 5e^2}{6} \epsilon^2 - \frac{36 - 15e + 15e^2 - 5e^3}{9} \epsilon^3. \quad (57)$$

Further, we find

$$k_3 = 1 - \epsilon - \frac{18 + 5e^2}{12} \epsilon^2 - \frac{126 - 30e + 45e^2 - 10e^3}{36} \epsilon^3, \quad (58)$$

$$k_3^{-1} = 1 + \epsilon + \frac{30 + 5e^2}{12} \epsilon^2 + \frac{270 - 30e + 75e^2 - 10e^3}{36} \epsilon^3. \quad (59)$$

The rotation for revolution of the periastron, by denoting with  $\omega$  its angular measure, is given, to order  $\epsilon^3$ , by

$$\begin{aligned} \Delta\omega &= 2\pi (k_3^{-1} - 1) = 2\pi\epsilon + 5\pi \left(1 + \frac{1}{6}e^2\right) \epsilon^2 \\ &\quad + 5\pi \left(3 - \frac{1}{3}e + \frac{5}{6}e^2 - \frac{1}{9}e^3\right) \epsilon^3. \end{aligned} \quad (60)$$

This formula agrees with the results obtained with the Poincaré-Lindstedt method and some of its equivalent modifications<sup>8,15</sup>. By denoting with  $P$  the anomalistic period, that is the time that elapses between two passages of the object at its perihelion expressed in days,

the average advance rate is

$$\begin{aligned} \dot{\omega}(\text{rad/d}) &= + \frac{\Delta\omega}{P} \approx \dot{\omega}_1 + \dot{\omega}_2 + \dot{\omega}_3 \\ &= + \frac{6\pi r^*}{a(1 - e^2)P} + \frac{15\pi r^{*2}(6 + e^2)}{2a^2(1 - e^2)^2P} \\ &\quad + \frac{15\pi r^{*3}(54 - 6e + 15e^2 - 2e^3)}{2a^3(1 - e^2)^3P}. \end{aligned} \quad (61)$$

What is the meaning of  $e$  in the solution  $w_n$  of order  $\epsilon^n$ ? In the zero-order solution  $w_0$ ,  $e$  is the eccentricity, but in the successive approximations it loses this characterization: the symbol  $e$  is simply a constant in the open interval  $(0,1)$  that one introduces in the initial conditions to express the shape of the orbit, and that coincides with the eccentricity of the osculating Kepler ellipse to the path followed by the planet when  $\varphi = 0$ .

### III. APPLICATIONS

In astrophysical applications Eq. (61), written in the form

$$f(r^*, a, e, \Delta\omega) = 0, \quad (62)$$

is an implicit relation between dynamical and orbital parameters characterizing the system under consideration, and it can be used to calculate any of them, once known the others. Thus, for example, in the solar system  $r^*$ ,  $a$ ,  $e$  are known, and we calculate  $\Delta\omega$ , the perihelion shift of the planetary orbits. It is also evident that a periastron advance is highly enhanced by small  $a$ 's (whence short periods) and high orbital eccentricities. A word of caution is needed here: we must consider the fact that a very large eccentricity can also mean a very small periastron distance  $a(1 - e)$ , and so we must stop precisely at the orbit that just grazes the surface of the star. The problem of detecting the motion of periastron (or of apostron) of some highly elliptic extrasolar planets has been considered by some authors<sup>9,17</sup>, and of course now the question is to determine the magnitude of the higher orders effects for some plausible orbital parameters. In the first column of Table I for comparison purposes we have inserted the data concerning the Sun-Mercury system, while the second and third columns are referred to two hypothetical exoplanets of a solar-mass star with great eccentricities and/or small radial distances. While is doubtful, given the particularly high levels of observational accuracy required, that it is actually possible to find planets with orbital characteristics fitted for this purpose, in meantime one can imagine a verification of the high-orders perihelion formula achieved by means of a man-made solar probe in a carefully planned celestial mechanics experiment. However, we believe that the higher order terms in Eq. (61) must be taken into account in all attempts to detect the Sun's Lense-Thirring effect on the perihelia of the inner planets in order to separate with improved



TABLE I: Mercury and two hypothetical exoplanets

System →	Sun/Mercury	Star/Alpha	Star/Beta
$M_\odot$	1.00	1.00	1.00
$\alpha$ (cm)	$1.475 \cdot 10^5$	$1.475 \cdot 10^5$	$1.475 \cdot 10^5$
$a$ (cm)	$5.791 \cdot 10^{12}$	$5.791 \cdot 10^{12}$	$8.788 \cdot 10^{10}$
$e$	0.2056	0.95	0.20
$\epsilon$	$1.038 \cdot 10^{-5}$	$1.038 \cdot 10^{-5}$	$1.038 \cdot 10^{-5}$
$P$ (d)	87.9	87.9	0.164
$\dot{\omega}_1$ (rad/d)	$5.703 \cdot 10^{-9}$	$5.602 \cdot 10^{-8}$	$2.001 \cdot 10^{-4}$
$\dot{\omega}_2$ (rad/d)	$1.097 \cdot 10^{-15}$	$1.262 \cdot 10^{-13}$	$2.652 \cdot 10^{-9}$
$\dot{\omega}_3$ (rad/d)	$2.46 \cdot 10^{-22}$	$2.873 \cdot 10^{-19}$	$4.098 \cdot 10^{-14}$
$\Delta\omega_1$ (arcsec/yr)	0.429	4.220	$1.514 \cdot 10^4$
$\Delta\omega_2$ (arcsec/yr)	$2.26 \cdot 10^{-10}$	$9.51 \cdot 10^{-6}$	$1.998 \cdot 10^{-1}$
$\Delta\omega_3$ (arcsec/yr)	$5.09 \cdot 10^{-17}$	$2.16 \cdot 10^{-11}$	$3.088 \cdot 10^{-6}$

accuracy these two important consequences of general relativity.

Some close binary systems are good subjects for a verification and an application of the formulas<sup>12</sup>. Before applying Eqs. (60), (61) to a concrete example, we recall that a generalization of Einstein's perihelion formula says that in a system of two non negligible spherical masses, the two-body problem, the gravitational parameter  $\mu$  means  $GM = G(m_1 + m_2)$ <sup>13</sup>. In this instance  $r^*$  could be loosely named the equivalent or the nominal gravitational radius of the system. This result is only the first-order approximation to a full relativistic treatment of the center of mass of two comparable bodies, a hitherto unresolved problem, but however we will fully exploit the effect of this approximation. For these systems, the semimajor axis and the total mass  $M$  are not directly determinable, while eccentricities, periods and periastron advance rate are, so we can use Eq. (62) to find  $r^*$ , once we have replaced  $a$  with the period  $P$  and the total mass  $M$  (the degeneracy of the mass is removed by measurement of another relativistic orbital characteristic, the Einstein parameter) using Kepler's third law in the form

$$a = \left( \frac{P^2 GM}{4\pi^2} \right)^{1/3} = \left( \frac{Pc}{2\pi} \right)^{2/3} r^{*1/3}. \quad (63)$$

To be precise, this law is not exactly true in a regime of strong fields and little distances, but it can be used harmlessly in a perturbative calculation for the systems we are considering within the same constraints we have set for the center of mass, that is a first-order approximation. Incidentally, this will free our results from other complex and still poor understood effects and, besides, this is in line with the regime of validity of the equation of motion which we have supposed. From Eqs. (61), (63)

we get

$$\dot{\omega} = \left( \frac{2\pi}{P} \right)^{5/3} \left( \frac{r^*}{c} \right)^{2/3} f(e) + \left( \frac{2\pi}{P} \right)^{7/3} \left( \frac{r^*}{c} \right)^{4/3} g(e) + \left( \frac{2\pi}{P} \right)^3 \left( \frac{r^*}{c} \right)^2 h(e), \quad (64)$$

where

$$f(e) = \frac{3}{(1 - e^2)},$$

$$g(e) = \frac{15(6 + e^2)}{2(1 - e^2)^2},$$

$$h(e) = \frac{15(54 - 6e + 15e^2 - 2e^3)}{4(1 - e^2)^3}.$$

To obtain directly the value of the mass  $M$  in units of the solar mass  $M_\odot$ , we can replace in Eq. (64) the ratio  $r^*/c$  with the product  $T_\odot M$ , where  $T_\odot \equiv GM_\odot/c^3$  is the mass of the Sun expressed in units of time. In the current literature the expression of  $\dot{\omega}$  lacks of the last two term of Eq. (64).<sup>12</sup> Inserting the values of  $e, P, \dot{\omega}$  determined for a given binary system, Eqs. (64), (61) can be numerically solved for  $r^*$  and  $M$  respectively, and will give the value of the gravitational radius of the system or the total mass. Once known  $r^*$ , we can complete the calculation and use Eq. (61) to compute the value of  $a$ . The value of  $\dot{\omega}$  that one determines is due, in general, to relativity plus extra classical terms, as the gravitational quadrupole moment induced by rotation and tidal deformations. But strongly self-gravitating objects as binary pulsars have a mass pointlike behavior, and thus the motion of their periastron must be entirely ascribed to relativity. Strictly speaking, since this bodies are rapidly spinning, one should consider also the Lense-Thirring effect or use the Kerr solution to the Einstein equation, but this is an argument for a further study.

We apply the theory to the double pulsar J0737-3039. Its orbital period is the smallest so far known for such an object, and it can be determined with great precision along with  $\dot{\omega}$ . For a such system cumulative effects add rapidly, and this allows a meaningful application of the formulas, despite the rather small eccentricity. In Table

TABLE II: Pulsar J0737-3039

$e$	0.0877775
$P$	0.10225156248(d) $\sim$ 8834.534991(s)
$\dot{\omega}$	16.89947 <sup>a</sup> (yr <sup>-1</sup> ) $\sim$ 9.346445651 $\cdot$ 10 <sup>-9</sup> (s <sup>-1</sup> )
$c$	2.99792458 $\cdot$ 10 <sup>10</sup> (cm/s)
$T_\odot$	4.925490947 $\cdot$ 10 <sup>-6</sup> (s)
$yr$	3.15576 $\cdot$ 10 <sup>7</sup> (s)

II we have indicated the determined<sup>12</sup> values of  $e, P, \dot{\omega}$  and, for the reader's convenience, some numerical values used in the calculations, while in Table III are indicated all parameters that can be derived through an application

TABLE III: Pulsar J0737-3039 derived parameters

To order:	$\epsilon$	$\epsilon^3$
$r^*$ (cm)	$3.82014 \cdot 10^5$	$3.8199525 \cdot 10^5$
$M(M_\odot)$	2.587075	2.586948
$a$ (cm)	$8.788391 \cdot 10^{10}$	$8.788680 \cdot 10^{10}$
$\epsilon$	$1.314166 \cdot 10^{-5}$	$1.314057 \cdot 10^{-5}$
$\dot{\omega}_1$	$16.89947^\circ \text{yr}^{-1}$	$16.89891408^\circ \text{yr}^{-1}$
$\dot{\omega}_2$	-	$0.00055589^\circ \text{yr}^{-1}$
$\dot{\omega}_3$	-	$0.00000002^\circ \text{yr}^{-1}$
$\dot{\omega}$	$16.89947^\circ \text{yr}^{-1}$	$16.89946999^\circ \text{yr}^{-1}$

of our formulas. In the first column, the computation are done to order  $\epsilon$ , assuming the validity of general relativity thorough the use of Einstein's precession formula  $\dot{\omega}_1$  of Eq. (61), while in the second column are inserted the values we have computed to order  $\epsilon^3$ , in particular how much of  $\dot{\omega}$  comes from  $\dot{\omega}_i$ ,  $i = 1, 2, 3$ . Since from the observational point of view the single higher-orders contributions to the precession rate are inextricably combined, and so hidden to a direct measurement, we can obtain an indirect verification turning to the main consequence of this approximation: a diminution by a small amount of the total mass of the system with respect to the currently accepted value, with the consequent redefinition of other parameters, in particular of  $a$ , and so we can refine the model of the system with one that fits the found variations.

Ever more accurate determinations of  $\dot{\omega}$  might enable, for this as for any other similar system, to test the usefulness of the third-order approximation to the periastron secular motion deduced from the equation of motion within the limits of approximation that we have introduced. However, the results found can be considered strictly true for the motion of a test body in the gravitational field of a central body of mass equivalent to the total mass of the binary system.

#### IV. APPENDIX

By simple considerations of powers and arguments, and with the aid of the subscript notation, we can quickly find the coefficient of the secular term  $H_n$  once known  $H_2, \dots, H_{n-1}$  and the orbit to order  $\epsilon^{n-1}$ . Here is a sketch of how we unearth from the right side of Eq. (54)

$$\epsilon(1 + w_2^2) - \frac{H_2}{e}w_2 = \epsilon(1 + w_2^2) - \epsilon(2A_1 + B_1)w_2, \quad (65)$$

the resonant term  $H_3 \cos k_{\mathcal{E}}\varphi$  of order  $\epsilon^3$ . We consider first the multinomial expression

$$\begin{aligned} \epsilon w_2^2 = & \epsilon[A_1 + A_2 + (e + E_1 + E_2) \cos k_{\mathcal{E}}\varphi \\ & + (B_1 + B_2) \cos 2k_{\mathcal{E}}\varphi + C_2 \cos 3k_{\mathcal{E}}\varphi]^2, \end{aligned} \quad (66)$$

together with the following algebraic and trigonometrical identities

$$\begin{aligned} \epsilon(a + b + c + \dots)^2 = & \epsilon a^2 + \epsilon b^2 + \epsilon c^2 + \dots \\ & \dots + 2\epsilon ab + 2\epsilon ac + \dots \\ & \dots + 2\epsilon bc + \dots, \end{aligned} \quad (67)$$

$$(\cos nk_{\mathcal{E}}\varphi)^2 = \frac{1}{2} \cos 2nk_{\mathcal{E}}\varphi + \frac{1}{2}, \quad (68)$$

$$a \cos(n+1)k_{\mathcal{E}}\varphi \cdot b \cos nk_{\mathcal{E}}\varphi = \frac{ab}{2} \cos k_{\mathcal{E}}\varphi + \dots \quad (69)$$

Here's the argument: in expanding Eq. (66) according to formula (67) we can omit to explicitly writing the cosines, since we know that each  $e, E$  multiplies  $\cos k_{\mathcal{E}}\varphi$ , so as each  $B$  and  $C$  multiplies  $\cos 2k_{\mathcal{E}}\varphi$  and  $\cos 3k_{\mathcal{E}}\varphi$  respectively. This way we can proceed rapidly by inspecting more concise expressions. Let us filter now Eq. (66) through the sieve represented by the constraints we have imposed to isolate the resonant term  $\sim \epsilon^3 \cos k_{\mathcal{E}}\varphi$ .

We observe first that the squared terms in Eq. (67) can be deleted because of Eq. (68) for  $n = 1, 2, 3$ . Next, we drop all double products of Eq. (67) in which the sum of the subscript indices is different from 2 and so, after multiplication by  $\epsilon$ , will survive only the terms of order  $\epsilon^3$ . Last, when we meet products of two cosines, we consider only those in which the arguments differ by one, and apply to them Eq. (69). At the end we obtain the following sum of coefficients of  $\cos k_{\mathcal{E}}\varphi$

$$2\epsilon e A_2 + \epsilon e B_2 + 2\epsilon A_1 E_1 + \epsilon E_1 B_1, \quad (70)$$

but the last two terms are erased by the  $\epsilon^3$ -coefficients arising from the rightmost expression of Eq. (65), which are the constants  $-2\epsilon A_1$  and  $-\epsilon B_1$  times the term  $E_1 \cos k_{\mathcal{E}}\varphi$  of  $w_2$ , and thus we finally get

$$H_3 = \epsilon e(2A_2 + B_2). \quad (71)$$

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